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Weak dependence of point processes and application to second order statistics

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We propose a general definition for weak dependence of point processes as an alternative to mixing definitions. We give examples of such weak dependent point processes for the families of Neyman Scott processes or Cox processes. For these processes, we consider the empirical estimator of the empty space function $F(r)$. Using the general setting of the weak dependence property, we show the Central Limit Theorem for a vector of such statistics with different r. This completes results establishing the Central Limit Theorem under the Poisson process hypothesis.

Keywords: Weak dependence; Cox process; Neyman-Scott process; Cluster statistics

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1. Introduction

We propose a new definition of weak dependence for point processes that is an alternative to the different mixing properties definitions. In the context of processes on $\mathbb{R}$ or $\mathbb{R}^d$, mixing properties of processes are very effective to prove asymptotic results for statistics that correspond to sum of functions of the variables of the process. The drawback is that mixing conditions are difficult to check even on simple models and they are not conservative under simple transformations. [7] defined weak dependence in a way that is much simpler to check on various models (see [4]). For point processes, mixing has been extended and studied on examples. [16] showed that strong mixing may be stable under clustering transformations but ϕ-mixing is not except under very restrictive assumptions. [12] proved regular and Brillinger mixing for some Markov point processes. Our aim is to extend weak dependence as they extended mixing to point processes, to show that some classes of point processes (Cox and Neyman-Scott processes) are weakly dependent under mild assumptions.

The second part of the paper is an application of weak dependence to the study of the asymptotic behaviour of a cluster statistic. This is not only an illustration of the theorical part; it has its own importance as providing a way to evaluate clustering tests

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powers. Let us recall the applications of cluster detection. Point processes are designed to model for locations of equivalent individuals on a map; this is the simplest kind of spatial data available, relevant in forestry for trees, in economics for shops or factories [17], or in biology for proteins fixed on cell membranes [14]. An introduction to the methods and applications can be found in [5, 13, 18]. Repartitions are roughly classified in three types: regular or overdispersed when the local density is everywhere constant, clustered if the points are grouped and random when the situation is in between. Regularity may procede from subjacent regularity in local conditions, as for example, when young trees were planted on a grid. It may also be the result of repulsive interactions between individuals as selection by concurrence for light. Clusters may derive from local better conditions or from positive interaction between individuals: for factories they may result from better local conditions in the access of a ressource; it may also come from mutual interest in a common formation of an adapted human capital; in a forest it may come either from better soil conditions or from spreading of seeds of a same mature individual.

Clusters and voids are also observed in random distribution with no interaction and no local heterogeneity as a homogeneous Poisson process sample. It is therefore essential to distinguish between clusters resulting from relevant interactions and inhomogeneity or from complete randomness. A wide literature is devoted to the question of the presence of significative clusters in spatial repartitions of a set of points. The first step was to propose a measurement of the clustering of the points. Many statistics have been defined, counting points in balls centered around fixed locations or points of the observed set. The Ripley statistic $\hat{K}(r)$ counts the number of couples of points with interdistance less than a fixed distance r. The distribution of cluster statistics under the homogeneous Poisson hypothesis was computed; [11] proposed the first multiscale Kolmogorov-Smirnov, Cramer-von Mises and chi-square goodness-of-fit tests based on the Ripley statistic for Poisson processes. [15] use an unbiased Ripley statistic to avoid edge effects. [10] propose a test that counts the number of points having exactly k neighbors at distance less than r. Less is known of the distribution of these statistics under counter-hypotheses: general results have been obtained by [8] for inhomogeneous Poisson processes. But there are still few results concerning the distribution of cluster statistics under dependent processes counter-hypotheses, such as Cox processes, Neyman Scott processes or Gibbs processes. To conclude on dependent point processes, recall that they find applications on the real line also. In extreme theory, the existence of clusters of values over a threshold has been studied under mixing conditions. Recently asks and bids arrivals in high-frequency trading have been modelised by Poisson processes [3]. But the question of clustering in time of the orders is a challenging problem that may be modelised with weak dependent point processes on the line.

In this paper, we define a general setting that characterizes weak dependence for point processes. The dependence is an extension of the weak dependence as defined by [7]. In Section 2, we set the definition and compare it with dependence at finite range. In section 3, we give conditions for Cox and Neyman-Scott processes to be weakly dependent in this sense. In section 4, we apply the condition of weak dependence to establish the central limit theorem of a vector of empty space function statistics computed for different distances. We conclude with perspectives of further work for other models and statistics.

2. Weak dependence for point processes

Here we give the definition of the generalization of weak dependence for point processes. First we recall the definition of dependence at finite range. Let X be a point process on $\mathbb{R}^d$. Let N denote the corresponding counting process. For convenience, we will use the L^1-distance in $\mathbb{R}^d$.

2.1. *Definition*

The simplest notion of weak dependence is the finite range dependence:

Definition 2.1 *X is a point process with dependence range r if for any compact subsets A and B of $\mathbb{R}^d$ such that the distance between A and B is more than r then $N(A)$ and $N(B)$ are independent.*

This definition is equivalent to the condition that for any square integrable functions f and g

$$\mathrm{Cov}\,(f(N(A), g(N(B))) = 0.$$

It will be relaxed in two ways, by considering bounded functions and bounding the covariance by a sequence decreasing to zero when the interdistance r tends to infinity. This leads to a definition of weak dependence that mimics the definition of [7] for time series and fields.

Definition 2.2 *X is a η-weakly dependent point process with rate $\eta(r)$ if for any integers u and v, any bounded real functions f on $\mathbb{R}^u$ and g on $\mathbb{R}^v$, any collection of non-intersecting compact sets $(A_i)_{i=1,\ldots,u}$ of $\mathbb{R}^d$ and any collection of non-intersecting compact sets $(B_j)_{j=1,\ldots,v}$ such that the distance between $\bigcup_{i=1\ldots,u} A_i$ and $\bigcup_{j=1\ldots,v} B_j$ is more than r then*

$$\begin{aligned} &|Cov\,(f(N(A_1),\ldots,N(A_u)),g(N(B_1),\ldots,N(B_v)))\,| \\ &\qquad\qquad \leq \psi\left((A_i)_{i=1,\ldots,u},(B_j)_{j=1,\ldots,v}\right)\|g\|_\infty\|f\|_\infty\eta(r). \end{aligned} \quad (1)$$

The set function ψ is in general $\sum_{i=1}^u 1 \vee \delta(A_i)^d + \sum_{j=1}^v 1 \vee \delta(B_j)^d$, where $\delta(K)$ is the diameter of the compact set K. But for Neyman-Scott processes, we will also use a different version of η-weak dependence, replacing ψ is by its square. We will use the short notation $\psi\left((A_i),(B_j)\right)$ in formulas.
Remarks:

- The counting variables $N(A_i)$ for compact sets A_i generate the σ-algebra generated by the point process. They are the reference variables adapted to the study of the dependence in the process. This covariance inequality will be used to prove the Central Limit Theorem for cluster statistics.
- The definition may be extended to processes with count expectations that are continuous with respect to another measure than the Lebesgue measure. The ψ function is then written with respect to this measure.

Note that the superimposition of two independent processes with dependence range r has dependence range r. The same property holds for weak dependent processes:

Proposition 2.1 *Consider two independent point processes X and Y. If these processes are η-weakly dependent with rates $\eta^X(r)$ and $\eta^Y(r)$ respectively then their superimposition is η-weakly dependent with rate $\eta^X(r)+\eta^Y(r)$.*

Proof. Define Z as the superimposition of X and Y. Denote N its counting process; denote

$$\begin{aligned} Z_A &= f(N(A_1),\ldots,N(A_u)) \\ Z_B &= g(N(B_1),\ldots,N(B_v)). \end{aligned}$$

Denote N^X and N^Y the counting processes of X and Y. Let $\mathbf{n}^x$ and $\mathbf{n}^y$ be elements of $\mathbb{N}^u$ and $\mathbf{p}^x$ and $\mathbf{p}^y$ be elements of $\mathbb{N}^v$. Let $\mathbf{n} = \mathbf{n}^x + \mathbf{n}^y$ and $\mathbf{p} = \mathbf{p}^x + \mathbf{p}^y$. Denote

$$\mathbb{P}_{XAB}(\mathbf{n}^x, \mathbf{p}^x) = \mathbb{P}\left((N^X(A_1), ..., N^X(A_u)) = \mathbf{n}^x, (N^X(B_1), ..., N^X(B_v)) = \mathbf{p}^x\right)$$
$$\mathbb{P}_{XA}(\mathbf{n}^x) = \mathbb{P}\left((N^X(A_1), ..., N^X(A_u)) = \mathbf{n}^x\right).$$

Because X and Y are independent:

$$\text{Cov}\,(Z_A, Z_B) = \sum_{\mathbf{n}^x,\mathbf{n}^y,\mathbf{p}^x,\mathbf{p}^y} f\,(\mathbf{n})\,g\,(\mathbf{p}) \times$$
$$(\mathbb{P}_{XAB}(\mathbf{n}^x, \mathbf{p}^x)\mathbb{P}_{YAB}(\mathbf{n}^y, \mathbf{p}^y) - \mathbb{P}_{XA}(\mathbf{n}^x)\mathbb{P}_{YA}(\mathbf{n}^y)\mathbb{P}_{XB}(\mathbf{p}^x)\mathbb{P}_{YB}(\mathbf{p}^y)).$$

The probabilities may be split into two parts:

$$\text{Cov}\,(Z_A, Z_B) =$$
$$\sum_{\mathbf{n}^x,\mathbf{p}^x} \mathbb{P}_{XAB}(\mathbf{n}^x, \mathbf{p}^x) \sum_{\mathbf{n}^y,\mathbf{p}^y} f\,(\mathbf{n})\,g\,(\mathbf{p})\,(\mathbb{P}_{YAB}(\mathbf{n}^y, \mathbf{p}^y) - \mathbb{P}_{YA}(\mathbf{n}^y)\mathbb{P}_{YB}(\mathbf{p}^y))$$
$$+ \sum_{\mathbf{n}^y,\mathbf{p}^y} \mathbb{P}_{YA}(\mathbf{n}^y)\mathbb{P}_{YB}(\mathbf{p}^y) \sum_{\mathbf{n}^x,\mathbf{p}^x} f\,(\mathbf{n})\,g\,(\mathbf{p})\,(\mathbb{P}_{XAB}(\mathbf{n}^x, \mathbf{p}^x) - \mathbb{P}_{XA}(\mathbf{n}^x)\mathbb{P}_{XB}(\mathbf{p}^x))$$

so that

$$|\text{Cov}\,(Z_A, Z_B)| \leq \sum_{\mathbf{n}^x,\mathbf{p}^x} \mathbb{P}_{XAB}(\mathbf{n}^x, \mathbf{p}^x)\|g\|_\infty\|f\|_\infty\psi\,((A_i), (B_j))\,\eta^Y(r)$$
$$+ \sum_{\mathbf{n}^y,\mathbf{p}^y} \mathbb{P}_{YA}(\mathbf{n}^y)\mathbb{P}_{YB}(\mathbf{p}^y)\|g\|_\infty\|f\|_\infty\psi\,((A_i), (B_j))\,\eta^X(r)$$
$$\leq \|g\|_\infty\|f\|_\infty\psi\,((A_i), (B_j))\,(\eta^Y(r) + \eta^X(r)).$$

■

As a direct corollary of Definition 2.2, we get a bound for products of count variables, when the count process itself is bounded (for example, when the total number of points is fixed).

Proposition 2.2 *Assume that X is a η-weakly dependent point process and that its count process is bounded by a constant M. Let $(A_i)_{i=1,\ldots,u}$ and $(B_j)_{j=1,\ldots,v}$ be two finite collections of compact sets separated by a distance r.*

$$\left|Cov\left(\prod_{i=1}^{u} N(A_i), \prod_{j=1}^{v} N(B_j)\right)\right| \leq \psi\,((A_i), (B_j))\,M^{u+v}\eta(r).$$

3. Examples

In this section, we show that some well known classes of processes are η-weakly dependent under relatively mild assumptions.

3.1. *A simple example on the line*

We first build a simple process that is weakly dependent in the sense of Proposition 2.2 but not strongly mixing. Recall the time series defined in [1] : let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a i.i.d. Bernoulli sequence with parameter $p = 1/2$. Define the stationary linear process

$$X_n = \sum_{i>0} 2^{-i}\varepsilon_{n-i};$$

Note that X_n is uniform in $[0, 1]$; one can use a binary representation $X_n = 0.\varepsilon_{n-1}...\varepsilon_{n-i}....$ The sequence (X_n) is weakly dependent (see [6]) but not regularly mixing, because X_0 is a deterministic function of X_n; deleting the n first binary digits of X_n gives X_0. Define the point process Z on the positive halfline with one point in each unit interval: P_n is the only point in $[n, n + 1)$ and is located at $n + X_n$. then this point process is not regularly mixing in the sense of theorem 2 of [12], because the location of P_n determines the location of P_0. We begin the study of the dependence of this process. Define $A(n, k)$ as a dyadic interval of length 2^{-k} in $[n, n+1]$ and $B(n+r, k')$ as a dyadic interval of length $2^{-k'}$ in $[n + r, n + r + 1]$; denote I the event $\{P_0 \in A\}$ and J the event $\{P_n \in B\}$.

- If $k' \le r$, the set of ε_i involved in the definition of I and J have no intersection. I and J are independent. Thus $\mathbb{P}(I \cap J) - \mathbb{P}(I)\mathbb{P}(J) = 0$.
- If $k' > r$, then the $k' - r$ first digits of the location of P_0 are known, so that $P_0 \in C$, where C is the dyadic interval of length $2^{-k'+r}$ corresponding to these first digits.
  - If $A \cap C = \emptyset$ then $\mathbb{P}(I \cap J) = 0$ so that $|\mathbb{P}(I \cap J) - \mathbb{P}(I)\mathbb{P}(J)| = 2^{-k}2^{-k'} \le 2^{-k}2^{-r}$.
  - If $A\cap C \ne \emptyset$ and $k > k'-r$ then $I \subset J$ and $k+r$ of the ε_i are fixed in the definition of $I\cap J$ so that $\mathbb{P}(I\cap J) \le 2^{-k-r}$ and $|\mathbb{P}(I\cap J)-\mathbb{P}(I)\mathbb{P}(J)| = 2^{-k}(2^{-r}-2^{-k'}) \le 2^{-k}2^{-r}$.
  - If $A \cap C \ne \emptyset$ and $k \le k' - r$ then $J \subset I$ and $\mathbb{E}(I \cap J) = \mathbb{P}(J) = 2^{-k'}$ so that $|\mathbb{P}(I \cap J) - \mathbb{P}(I)\mathbb{P}(J)| = 2^{-k'}(1 - 2^{-k}) \le 2^{-k'} \le 2^{-k-r}$.

Consider now two collections $(A(k_i, n_i))_{i=1,...,u}$ and $(B(k'_j, n'_j) =_{j=1,...,v)}$ of dyadic intervals with $n_1 < \ldots < n_u < n'_1 + r < \ldots < n'_v$. We give a bound to $c_{A,B} = \mathrm{Cov}\left(\prod_{i=1}^u N(A_i), \prod_{i=1}^v N(B_j)\right)$.

- If for all j, $n'_j - k'_j > n_u$, $\prod_{i=1}^u N(A_i)$ and $\prod_{i=1}^v N(B_j)$ are independent.
- If not, define j_m as the index corresponding to the minimum value of $n'_j - k'_j$. Then the $n'_{j_m} - k'_{j_m} - n_u$ first digits of the location of P_{n_u} are known, so that $P_{n_u} \in C$, where C is the corresponding dyadic interval of length $2^{n'_j-k'_j-n_u}$.
  - If $A(k_u, n_u) \cap C = \emptyset$ then $\mathbb{E}(\prod_{i=1}^u N(A_i)\prod_{i=1}^v N(B_j)) = 0$ so that $|c_{A,B}| = 2^{-k_u}2^{-k'_j} \le 2^{-k_u}2^{-r}$.
  - If $A(k_u, n_u) \cap C \ne \emptyset$ and $k_u > k'_j - n'_j + n_u$ then $k_u + n'_j - n_u$ of the ε_i are fixed in the definition of $(\prod_{i=1}^u N(A_i)\prod_{i=1}^v N(B_j))$ so that $|\mathbb{E}(\prod_{i=1}^u N(A_i)\prod_{i=1}^v N(B_j))| \le 2^{-k_u-n'_j+n_u}$ and $|\mathbb{E}(\prod_{i=1}^u N(A_i))\mathbb{E}(\prod_{i=1}^v N(B_j))| \le 2^{-k_u-k'_j}$. We get $|c_{A,B}| \le 2.2^{-k_u}2^{-r}$.
  - If $A(k_u, n_u)\cap C \ne \emptyset$ and a $k_u \le k'_j - n'_j + n_u$ then k'_j of the ε_i are fixed in the definition of $(\prod_{i=1}^u N(A_i)\prod_{i=1}^v N(B_j))$ and so that $|\mathbb{E}(\prod_{i=1}^u N(A_i)\prod_{i=1}^v N(B_j))| \le 2^{-k'_j}$. We get $|c_{A,B}| \le 2^{-k'_j}(1 + 2^{-k_u}) \le 2.2^{-k_u}2^{-r}$.

From this, we get proposition 2.2 with $M = 1$ and $\eta(r) = 2^{1-r}$.

3.2. *Neyman-Scott process*

Recall that a Neyman-Scott process is defined as follows. First, germs of the process are drawn following a homogeneous Poisson process. Then each germ is replaced by its offspring. The offspring process is an inhomogeneous point process centered on the corresponding germ. The offsprings of the different germs are identically independently distributed (up to translation to the parent germ).

PROPOSITION 3.1 *A Neyman-Scott process with an offspring process with compact support K in the ball $B(0, r/2)$ is dependent with range r.*

Proof. Note that the offspring process has dependence range r, because if two sets have interdistance larger than r, one of them is constantly void. Assume that the two collections of sets (A_i) and (B_j) have interdistance larger than r. Denote $X_A = f(N(A_1), \ldots, N(A_u))$ and $X_B = g(N(B_1), \ldots, N(B_v))$. Define $A^K = \{x : \text{ there exist } y \in \cup_i A_i,\ z \in K \text{ such that } x = y + z\}$ and $B^K = \{x : \text{ there exist } y \in \cup_i B_i,\ z \in K \text{ such that } x = y + z\}$. Then, X_A depends only on offsprings of germs located in A^K and X_B depends only on offsprings of germs located in B^K. As these two sets do not intersect the corresponding offspring points comes from independent populations so that Cov $(X_A, X_B) = 0$. ■

Assume now that the offspring process is an inhomogeneous Poisson process with an isotropic intensity $\rho(r)$. Define $p_O(r)$ as the probability that the offspring process has no points outside the ball of radius r:

$$p_O(r) = \exp\left(-\int_r^\infty a_{d-1}\rho(s)s^{d-1}ds\right)$$

where $a_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$ is the measure of the d-dimensional unit sphere. Define $P(\delta, r)$ as the probability that for a given ball $B(0, \delta)$, at least one point in $B(0, \delta)$ comes from out of the ball $B(0, \delta + r)$. Then

$$\begin{aligned} P(\delta, r) &\le \int_r^\infty \lambda a_{d-1}(1 - p_O(s))(s + \delta)^{d-1}ds \\ &\le 2^{d-1}\lambda a_{d-1}\int_r^\infty (1 - p_O(s))(s \vee \delta)^{d-1}ds \\ &\le 2^{d-1}\lambda a_{d-1}\left(\delta^{d-1}\int_r^\infty (1 - p_O(s))ds + \int_r^\infty (1 - p_O(s))s^{d-1}ds\right) \\ &\le (1 \vee \delta^d)\lambda f_1(r) + \lambda f_2(r), \end{aligned}$$

with $f_1(r) = 2^{d-1}a_{d-1}\int_r^\infty(1 - p_O(s))ds$ and $f_2(r) = 2^{d-1}a_{d-1}\int_r^\infty(1 - p_O(s))s^{d-1}ds$.

PROPOSITION 3.2 *If the offspring process is an inhomogeneous Poisson process such that $f_2(r)$ tends to zero, then the corresponding Neyman-Scott process is η-weakly dependent. Its rate $\eta(r)$ is less than $\lambda(f_1(r/2)+f_2(r/2))$, where λ is the intensity of the germ process.*

Proof. Denote $X_A = f(N(A_1), \ldots, N(A_u))$ and $X_B = g(N(B_1), \ldots, N(B_v))$. Let $\mathbf{m} = (F_i)_{i\in\mathbb{N}}$ be the locations of the germs, then

$$\text{Cov }(X_A, X_B) = \mathbb{E}\text{ Cov }(X_A, X_B|\mathbf{m}) + \text{Cov }(\mathbb{E}(X_A|\mathbf{m}), \mathbb{E}(X_B|\mathbf{m}))\,.$$

Conditionally to the germ process, the process is a countable superimposition of inho-

mogeneous Poisson processes, that is, an inhomogeneous Poisson process itself so that the first covariance is zero.
Denote $Y_A = \mathbb{E}(X_A|\mathbf{m})$ and $Y_B = \mathbb{E}(X_A|\mathbf{m})$. We distinguish points that come from close germs from the others. As $\delta(A_i)$ is the diameter of A_i, there exists a ball of radius $\delta(A_i)/2$ that contains A_i. Inflating this ball to a radius $(\delta(A_i)+r)/2$ gives a ball A_i^r containing the $r/2$-neighborhood of A_i. Let $A_r = \cup_{i=1\dots u} A_i^r$ and $B_r = \cup_{j=1\dots v} B_j^r$. Define $N^{loc}(A_1)$ as the number of points that are offsprings of germs inside A_r and $Y_A^{loc} = f(N^{loc}(A_1), \dots, N^{loc}(A_u))$:

$$|Y_A - Y_A^{loc}| \le 2\|f\|_\infty \sum_{i=1}^{u} \mathrm{I}\{N(A_i) \ne N^{loc}(A_i)\}.$$

Define similarly Y_B^{loc} with respect to Y_B, then:

$$|Y_B - Y_B^{loc}| \le 2\|g\|_\infty \sum_{i=1}^{v} \mathrm{I}\{N(B_i) \ne N^{loc}(B_i)\}.$$

Define Z_A, Z_A^{loc}, Z_B and Z_B^{loc} as the recentered variables corresponding to Y_A, Y_A^{loc}, Y_B and Y_B^{loc}. As $\mathbb{E}\left(\mathrm{I}\{N(A_i) \ne N^{loc}(A_i)\}\right) \le P(\delta(A_i), r/2)$, we get

$$\mathbb{E}|Z_A - Z_A^{loc}| \le 4\|f\|_\infty \sum_{i=1}^{u} P(\delta(A_i), r/2)$$
$$\mathbb{E}|Z_B - Z_B^{loc}| \le 4\|g\|_\infty \sum_{i=1}^{v} P(\delta(B_i), r/2)$$

so that

$$\begin{aligned}
&|\mathrm{Cov}\ (Y_A, Y_B) - \mathrm{Cov}\ \left(Y_A^{loc}, Y_B^{loc}\right)| \\
&\qquad \le \left|\mathbb{E}\left(Z_A(Z_B - Z_B^{loc})\right)\right| + \left|\mathbb{E}\left((Z_A - Z_A^{loc})Z_B^{loc}\right)\right| \\
&\qquad\qquad \le 4\|g\|_\infty \|f\|_\infty \left(\sum_{i=1}^{u} P(\delta(A_i), r/2) + \sum_{i=1}^{v} P(\delta(B_i), r/2)\right).
\end{aligned}$$

But Y_A^{loc} and Y_B^{loc} are independent so that

$$\begin{aligned}
|\mathrm{Cov}\ (Y_A, Y_B))| &\le 4\|g\|_\infty \|f\|_\infty \left[\psi((A_i),(B_i)) f_1(r/2) + (u+v) f_2(r/2)\right] \\
&\le 4\|g\|_\infty \|f\|_\infty \psi((A_i),(B_j)) \left(f_1(r/2) + f_2(r/2)\right).
\end{aligned}$$

and

$$|\mathrm{Cov}\ (X_A, X_B)| \le 4\|g\|_\infty \|f\|_\infty \psi((A_i),(B_j)) \left(f_1(r/2) + f_2(r/2)\right).$$

■

Now we can mix the weak dependence effect coming from possible superimposition of independent offspring populations as in the preceding process and a weak dependence inherited from the weak dependence of the offspring process itself.

PROPOSITION 3.3 *If the offspring process is η-weakly dependent and its intensity is isotropic and such that $f_2(r)$ tends to zero, then the corresponding Neyman-Scott process is η-weakly dependent. Its rate is*

$$\eta(r) \leq \lambda \left(\frac{2^{d-1} a_{d-1}}{2d} (2^d + r^d) \eta^O(r) + 8 \left(f_1(r/2) + f_2(r/2) \right) \right),$$

where λ is the intensity of the germ process and $\eta^O(r)$ is the rate of the offspring process. The set function ψ has to be replaced by ψ^2.

This leads to a new class of stationary processes that can be used for modelling two scales effects. Assume that the offspring process is repulsive. Then we observe clusters with size depending on the range of the intensity of the offspring process and repulsive effects at a lesser range between points of the offspring.

Proof. The only difference with the preceding result comes from the first term, for which we use the weak dependence property. Let R be a distance and U_R be the union of balls $(A_i)^R$ or $(B_j)^R$ each covering the R-neighborhood of the corresponding compact set in the collection (A_i) or (B_j). We consider the process as the independent superimposition of the offspring processes of germs in U_R and the offspring processes of germs outside U_R. Let N be the number of germs that are in U_R and $\mathbf{u} = (u_i)_{i,\ldots,N}$ their locations. Let $\mathbf{v} = (v_i)_{i \in \mathbb{N}}$ be the collection of germs that are outside U_R. Define Y_A as the value of X_A obtained when erasing all the offsprings of the germs of $\mathbf{v}$. then:

$$\text{Cov}\ (X_A, X_B) = \mathbb{E}\ \text{Cov}\ (X_A, X_B | \mathbf{u}, \mathbf{v}) + \text{Cov}\ (\mathbb{E}(X_A | \mathbf{u}, \mathbf{v}), \mathbb{E}(X_B | \mathbf{u}, \mathbf{v}))\,.$$

$$|\text{Cov}\ (X_A, X_B | \mathbf{u}, \mathbf{v}) - \text{Cov}\ (Y_A, Y_B)\,| \leq 4 \|g\|_\infty \|f\|_\infty \psi((A_i), (B_j))\, (f_1(R) + f_2(R))).$$

Conditionally to the germ process, Y_A and Y_B result from is the superimposition of N independent η-weakly dependent processes, so that, following proposition 2.1:

$$|\text{Cov}\ (Y_A, Y_B)\,| \leq N \|g\|_\infty \|f\|_\infty \psi((A_i), (B_j)) \eta^O(r).$$

Integrating with respect to the locations u_i keeps the bound unchanged. Integrating with respect to N gives

$$|\mathbb{E}\ \text{Cov}\ (Y_A, Y_B)| \leq \|g\|_\infty \|f\|_\infty \psi((A_i), (B_j)) \lambda m(U_R) \eta^O(r).$$

Note that

$$m(A_i^R) \leq \frac{2^{d-1} a_{d-1}}{d} \left(\delta(A_i)^d + R^d \right)$$

and

$$\begin{aligned} m(U_R) &\leq \frac{2^{d-1} a_{d-1}}{d} \left(\sum_{i=1}^{u} \delta(A_i)^d + \sum_{i=1}^{v} \delta(B_i)^d + (u+v) R^d \right) \\ &\leq \frac{2^{d-1} a_{d-1}}{d} \psi((A_i), (B_j))(1 + R^d). \end{aligned}$$

$$|\mathbb{E}\ \mathrm{Cov}\ (X_A, X_B|\mathbf{u}, \mathbf{v})| \leq \lambda \|g\|_\infty \|f\|_\infty \psi^2((A_i), (B_j)) \\ \times \left(\frac{2^{d-1}a_{d-1}}{d}(1+R^d)\eta^O(r) + 4\left(f_1(R)+f_2(R)\right)\right).$$

The bound of the second term is unchanged:

$$|\mathrm{Cov}\ (\mathbb{E}(X_A|\mathbf{u}, \mathbf{v}), \mathbb{E}(X_B|\mathbf{u}, \mathbf{v}))\,| \leq 4\|g\|_\infty \|f\|_\infty \psi((A_i), (B_j))\,(f_1(r/2)+f_2(r/2))).$$

so that

$$|\mathrm{Cov}\ (X_A, X_B)\,| \leq \lambda \|g\|_\infty \|f\|_\infty \psi^2((A_i), (B_j)) \\ \times \left(\frac{2^{d-1}a_{d-1}}{d}(1+R^d)\eta^O(r) + 4\left(f_1(R)+f_2(R)+f_1(r/2)+f_2(r/2)\right)\right).$$

A rough bound may be obtained by choosing $R = r/2$, leading to the result. But R may be also be chosen less than this value to balance the first and second term. ■

3.3. *Cox process*

We consider Cox processes whose random measure of intensity is continuous with respect to the Lebesgue measure of the plane:

DEFINITION 3.1 *Consider a stationary positive bounded Lipschitz field Z on $\mathbb{R}^d$. The Cox process is the point process that is conditionally to Z an inhomogeneous Poisson process of intensity Z.*

The weak dependence property is inherited from the weak dependence property of the intensity field (see [4]): recall that a field is η-weakly dependent with rate $\eta(r)$ if for any integers u and v, any bounded real functions f on $(\mathbb{R}^d)^u$ and g on $(\mathbb{R}^d)^v$ that are Lipschitz with respect to the $\mathbb{L}^1$-norm, any sequences $(x_i)_{i=1\ldots,u}$ and $(y_j)_{i=1\ldots,v}$ of $\mathbb{R}^d$ such that the distance between the x_i's and the y_j's is more than r then :

$$|\mathrm{Cov}\ (f(x_1,\ldots,x_u), g(y_1,\ldots,y_v))\,| \leq (u\|g\|_\infty \mathrm{Lip}\,(f) + v\|f\|_\infty \mathrm{Lip}\,(g))\eta(r).$$

PROPOSITION 3.4 *A Cox process is η-weakly dependent with rate $\eta(r)$ if its intensity field is bounded, Lipschitz and η-weakly dependent with rate $\eta^Z(r)$, and $\eta(r) \leq (4K/a^d)\eta^Z(r)$, where K is the Lipschitz constant of Z.*

Proof. Define X_A and X_B as in the preceding section, then:

$$\mathrm{Cov}\ (X_A, X_B) = \mathbb{E}\ \mathrm{Cov}\ (X_A, X_B|Z) + \mathrm{Cov}\ (\mathbb{E}(X_A|Z), \mathbb{E}(X_B|Z))\,.$$

Conditionally to Z, X is a Poisson process. The first term is zero because the counting processes $N(A_i)$ and $N(B_j)$ are independent for a Poisson process as soon as the sets are non intersecting. Then

$$\begin{aligned}\mathbb{E}(X_A|Z) &= \sum_{n_1,\ldots,n_u=0}^{\infty} f(n_1,\ldots,n_u)\prod_{i=1}^{u}\mathbb{P}(N(A_i)=n_i)\\ &= \sum_{n_1,\ldots,n_u=0}^{\infty} f(n_1,\ldots,n_u)\prod_{i=1}^{u} f_{n_i}(Z(A_i)).\end{aligned}$$

where $f_n(x) = e^{-x}x^n/n!$. Define

$$F(x_1,\ldots,x_u) = \sum_{n_1,\ldots,n_u=0}^{\infty} f(n_1,\ldots,n_u)\prod_{i=1}^{u} f_{n_i}(x_i).$$

Then $\mathbb{E}(X_A|Z) = F(Z(A_1),\ldots,Z(A_u))$. We first prove that F is Lipschitz. Recall that $f_n(x)$ is a bounded and C^1 function with derivative $f_n'(x) = e^{-x}(nx^{n-1} - x^n)/n!$. Then

$$\begin{aligned}\left|\frac{\partial F}{\partial x_1}(x_1,\cdots,x_u)\right| &\le \sum_{n_1,\ldots,n_u=0}^{\infty} |f(n_1,\ldots,n_u)|\\ &\quad\times (n_1x_1^{n_1-1} + x_1^{n_1})\frac{e^{-x_1}}{n_1!}\prod_{i=2}^{u} f_{n_i}(x_i)\\ &\le \|f\|_\infty \sum_{n_1,\ldots,n_u=0}^{\infty} (n_1x_1^{n_1-1} + x_1^{n_1})\frac{e^{-x_1}}{n_1!}\prod_{i=2}^{u} f_{n_i}(x_i)\\ &\le \|f\|_\infty \sum_{n_1=0}^{\infty} (n_1x_1^{n_1-1} + x_1^{n_1})\frac{e^{-x_1}}{n_1!}\prod_{i=2}^{u}\sum_{n_i=0}^{\infty} f_{n_i}(x_i)\\ &\le 2\|f\|_\infty\end{aligned}$$

and the same bound is true with the other partial derivatives so that the Lipschitz coefficient of F is less than $2\|f\|_\infty$. Let $(A_{i,j})_{j\in J_i}$ be a collection of partitions of A_i such that the diameters of the $A_{i,j}$ are less than ε. Fix a point $x_{i,j}$ in each of the $A_{i,j}$. Then

$$Z(A_i) = \int_{A_i} Z(x)dx = \bar{Z}(A_i) + R_i,$$

with $|R_i| \le m(A_i)K\varepsilon$ and

$$\bar{Z}(A_i) = \sum_{j\in J_i} m(A_{i,j})Z(x_{i,j}).$$

Then

$$\begin{aligned}|\text{Cov }(X_A, X_B)| \le& \left|\text{Cov }\left(F(\bar{Z}(A_1),\cdots,\bar{Z}(A_u)), G(\bar{Z}(B_1),\cdots,\bar{Z}(B_u))\right)\right|\\ &+\|g\|_\infty\mathbb{E}\left|F(Z(A_1),\cdots,Z(A_u)) - F(\bar{Z}(A_1),\cdots,\bar{Z}(A_u))\right|\\ &+\|f\|_\infty\mathbb{E}\left|G(Z(B_1),\cdots,Z(B_u)) - G(\bar{Z}(B_1),\cdots,\bar{Z}(B_u))\right|.\end{aligned}$$

Write $F(\bar{Z}(A_1), \cdots, \bar{Z}(A_u)) = \bar{F}(((x_{i,j})_{i=1,..u;j\in J_i})$ then $\mathrm{Lip}\,\bar{F} \leq 2\|f\|_\infty \varepsilon^d K$ and the number of $x_{i,j}$ is less than $\varepsilon^{-d}\sum m(A_i)$.

$$\begin{aligned}
&\left|\mathrm{Cov}\left(F(\bar{Z}(A_1), \cdots, \bar{Z}(A_u)), G(\bar{Z}(B_1), \cdots, \bar{Z}(B_u))\right)\right| \\
&\qquad \leq 2\|g\|_\infty \|f\|_\infty \left(\sum_{i=1}^{u} m(A_i) + \sum_{j=1}^{v} m(B_j)\right) K\eta^Z(r) \\
&\qquad\qquad \leq 2\|g\|_\infty \|f\|_\infty \psi((A_i),(B_j))(K/a^d)\eta^Z(r)
\end{aligned}$$

$$\mathbb{E}\left|F(Z(A_1), \cdots, Z(A_u)) - F(\bar{Z}(A_1), \cdots, \bar{Z}(A_u))\right| \leq 2\|f\|_\infty K\varepsilon \sum_{i=1}^{u} m(A_i)$$

Choosing $\varepsilon < \eta_r$ gives the result. ■

4. Empty space function

Recall that the empty space function $F(r)$ is the probability that a fixed point of the space has no sample point at distance less than r. This function is used alone or together with the $G(r)$ function corresponding to the probability that the nearest neighbor of a point in the sample is at distance greater than r . These two functions are known and equal in the case of a homogeneous Poisson process. The empirical empty space function and the empirical nearest neighbor function are summary statistics used to detect regularity $(F(r) < G(r))$ or clustering $(G(r) < F(r))$ for a small scale r (see [5] and [9]). In the section, we prove the Central Limit Theorem for the empirical estimator $F_n(r)$ under the weak dependence assumption.

4.1. *Assumptions and notations*

We assume that we observe samples of a η-weakly dependent process on the set $A_n = [0,n]^d$ and that the size n goes to infinity. Fix a grid step $1/k$, where $k > 0$ is an integer. Divide A_n in $(nk)^d$ cube of side $1/k$ and consider the regular grid G formed by the centers $(m_i)_{i\in\{0,\ldots,nk\}^d}$ of these cubes. Given a distance $r < 1/(2k)$, the statistic counts the number of points of the grid that have no sample points at distance $\mathbb{L}^1$ less than r:

$$\widehat{F}_n(k,r) = \frac{1}{(nk)^d} \sum_{i\in\{0,\ldots,nk-1\}^d} \mathrm{I}\{N(C_{i,r}) = 0\},$$

where $C_{i,r}$ is a cube centered in m_i with half side r. Note that this statistic is a count, so that it is additive with respect to sets. Define $\eta_i(r) = \mathrm{I}\{N(C_{i,r}) = 0\}$ and $\zeta_i(r) = \eta_i(r) - \mathbb{E}(\eta_i(r))$ then

$$\widehat{F}_n(r) - \mathbb{E}(\widehat{F}_n(r)) = \frac{1}{(nk)^d} \sum_{i\in\{0,\ldots,nk-1\}^d} \zeta_i(r).$$

4.2. *Central Limit Theorem for the empirical empty space function*

We show that a normalized vector of empirical empty space function for different r converges in distribution to a normal vector. For this statistic is a M-statistic, we can apply results on sums of weak dependent variables to prove the following CLT:

THEOREM 4.1 *Assume that X is a η-weakly dependent point process*

- $\eta(r) \leq C$, *if* $r < 1$
- $\eta(r) \leq Cr^{-\gamma}$, *if* $r \geq 1$,

with $C > 1$ and $\gamma > 5d/3$. Let ℓ be an integer, $0 < r_1 < \ldots < r_\ell < \frac{1}{2k}$ a set of reals, and denote $\mathbf{F}_n = (\widehat{F}_n(r_1), \ldots, \widehat{F}_n(r_\ell))$:

$$(nk)^{d/2}(\mathbf{F}_n - \mathbb{E}(\mathbf{F}_n)) \longrightarrow \mathcal{N}(0, \Sigma)$$

where $\Sigma_{s,t} = \sum_{i \in \mathbb{Z}^d} Cov(\zeta_0(r_s), \zeta_i(r_t))$.

Proof. We prove the theorem for the special case where $k = 1$. Then for $k > 1$, we consider the rescaled process where all distances are multiplied by k and observe it on $A'(n) = [0, n']^d$ with $n' = kn$. The empty space empirical function $\widehat{F}_n(k, r)$ for the original process X is equal to $\widehat{F}_{n'}(1, kr)$ for the rescaled process X' and X' is η-weak dependent for $\eta'(r) = \eta(kr)$. Defining $C' = \max(1, Ck^{-\gamma})$, X' satisfies the weak dependence property and we may apply the result for $k = 1$.
Fix k=1. We prove first that the series defining the covariance matrix is convergent. Note that if two grid points m_i and m_j have interdistance $j > 1$, the corresponding variables $\zeta_i(r_s)$ and $\zeta_j(r_t)$ are counting measures on cubes with interdistance $j - r_s - r_t$ and

$$\text{Cov}\left(\zeta_i(r_s), \zeta_i(r_t)\right) \leq 2\eta(j - r_s - r_t) \leq 2C(j-1)^{-\gamma}$$

There are at most $2d(2j+1)^{d-1}$ grid points m_i at distance j from the grid point m_0, so that

$$\left| \sum_{i \in \mathbb{Z}^d} \text{Cov}\left(\zeta_0(r_s), \zeta_i(r_t)\right) \right| \leq 4dC3^{d-1} + 4dC \sum_{j>1} (2j+1)^{d-1}(j-1)^{-\gamma} < \infty.$$

We show that any linear combination of the $\widehat{F}_n(r_t)$ is asymptotically normal. Let $(\lambda_1, \ldots \lambda_\ell)$ be a vector of real coefficients and

$$L_n = \sum_{t=1}^{\ell} \lambda_t \widehat{F}_n(r_t).$$

We use the Bernstein blocks technique. Set $p = [n^\alpha]$ and $q = [n^\beta]$ with $0 < \beta < \alpha < 1$ to be chosen later. We divide the grid G into cubic grids of cardinal p^d. These grids are separated by gaps of q grid points. Let a be the integer quotient of n by $p + q$. To each multi-index i in $\{0, \ldots, a\}^d$ corresponds a cubic grid $P_{i,n}$ and we define $Q_n = A_n \backslash \bigcup_i P_{i,n}$

as the set of grid points that are in none of the $P_{i,n}$'s. For each set $P_{i,n}$ and Q_n, we define:

$$u_{i,n} = \frac{1}{n^{d/2}} \sum_{m_i \in P_{i,n}} \sum_{t=1}^{\ell} \lambda_t \zeta_i(r_t)$$

$$v_n = \frac{1}{n^{d/2}} \sum_{m_i \in Q_n} \sum_{t=1}^{\ell} \lambda_t \zeta_i(r_t).$$

then

$$n^{d/2}(L_n - \mathbb{E}L_n) = \sum_{i \in \{0,\dots,a\}^d} (u_{i,n} - \mathbb{E}u_{i,n}) + v_n - \mathbb{E}v_n.$$

We show that the sum of the $u_{i,n}$ converges in distribution to a Gaussian variable and that the other term is negligible in $\mathbb{L}^2$. We check the conditions of the following CLT from [2].

THEOREM 4.2 *Let $(z_{i,n})_{0 \le i \le k(n)}$ be an array of random variables satisfying*

(1) There exists $\delta > 0$ such that $\sum_{i=0}^{k(n)} \mathbb{E}|z_{i,n}|^{2+\delta}$ tends to 0 as n tends to infinity,
(2) $\sum_{i=0}^{k(n)} Var\, z_{i,n}$ tends to σ^2 as n tends to infinity,
(3) $T(n) = \sum_{j=1}^{k(n)} \left| Cov(e^{it(z_{0,n}+\dots+z_{j-1,n})}, e^{itz_{j,n}}) \right|$ tends to 0 as n tends to infinity.

then $\sum_{i=0}^{k(n)} z_{i,n}$ tends in distribution to $\mathcal{N}(0,\sigma^2)$ as n tends to infinity.

To check Condition 1, we compute the fourth order moment of $u_{1,n} - \mathbb{E}u_{1,n}$. For each m_i in $P_{1,n}$, denote for short $\zeta_i = \sum_{t=1}^{d} \lambda_t \zeta_i(r_t)$. Define as in [6],

$$A_2 = \sum_{i_1, i_2 \in \{0,\dots,p-1\}^d} |\mathbb{E}(\zeta_{i_1}\zeta_{i_2})|, \;\; A_4 = \sum_{i_1,\dots i_4 \in \{0,\dots,p-1\}^d} |\mathbb{E}(\zeta_{i_1}\zeta_{i_2}\zeta_{i_3}\zeta_{i_4})|,$$

$$\mathbb{E}((u_{i,n} - \mathbb{E}u_{i,n})^4) \le \frac{A_4}{n^{2d}} \; .$$

For each multi-index $\mathbf{i} = (i_1, \dots i_4)$, we define the gap r as the largest L^1-distance obtained by separating the grid points m_{i_1}, m_{i_2}, m_{i_3} and m_{i_4} into two groups. This defines a partition of the indices in $\mathbf{i}$ into two non void sets. For the two members of this partition, we denote $\Pi(\mathbf{i},1)$ and $\Pi(\mathbf{i},2)$ the product of the corresponding variables ζ. Then we sort the multi-indices by their gap. $G_{j,4}$ is the set of multi-indices $\mathbf{i}$ with gap j. Note that the gap is less than p.

$$\begin{aligned} A_4 &\le \sum_{i \in \{0,\dots,p-1\}^d} \mathbb{E}(\zeta_i^4) + \sum_{j=1}^{p} \sum_{\mathbf{i} \in G_{j,4}} \mathrm{Cov}\,(\Pi(\mathbf{i},1), \Pi(\mathbf{i},2)) + |\mathbb{E}(\Pi(\mathbf{i},1))\mathbb{E}(\Pi(\mathbf{i},2))| \\ &\le p^d \mathbb{E}(\zeta_1^4) + \sum_{j=1}^{p} \sum_{\mathbf{i} \in G_{j,4}} |\mathrm{Cov}\,(\Pi(\mathbf{i},1), \Pi(\mathbf{i},2))| + A_2^2 \end{aligned}$$

We decompose A_2 the same way, the gap being the L^1-distance between the grid points

m_{i_1} and m_{i_2}:

$$A_2 = \sum_{i\in\{0,\dots,p-1\}^d} \mathbb{E}(\zeta_i^2) + \sum_{i_1\neq i_2\in\{0,\dots,p-1\}^d} |\mathrm{Cov}(\zeta_{i_1},\zeta_{i_2})| \leq p^d\mathbb{E}(\zeta_1^2) + \sum_{j=1}^{p}\sum_{\mathbf{i}\in G_{j,2}} |\mathrm{Cov}(\zeta_{i_1},\zeta_{i_2})|$$

Now we evaluate a bound for the second term; for $\mathbf{i}\in G_{j,2}$,

$$|\mathrm{Cov}(\zeta_{i_1},\zeta_{i_2})| \leq 2C(j-1)^{-\gamma}.$$

The cardinality of $G_{j,2}$ is bounded as follows: consider one of the p^d grid points m_i. There are at most $2d(2j+1)^{d-1}$ grid points separated from m_i by a gap equal to j. So

$$A_2 \leq p^d\mathbb{E}(\zeta_1^2) + 4Cdp^d\sum_{j=1}^{p}(2j+1)^{d-1}(j-1)^{-\gamma},$$

so that we get $A_2 = O\left(p^d\right)$.
Similarly, to build a multi-index $\mathbf{i}$ in $G_{j,4}$, take the first point among the p^d possible and the second point separated by a gap j (at most $2(2j+1)^d$ possibilities); the third point has to be chosen at a distance less than j from the two preceding points (at most $2(2j+1)^d$ possibilities) and the fourth at a distance less than j from the three preceding points (at most $3(2j+1)^d$ possibilities). So the cardinality of $G_{j,4}$ is bounded by $12dp^d(2j+1)^{3d-1}$. For each of these indices, $|\mathrm{Cov}(\Pi(\mathbf{i},1),\Pi(\mathbf{i},2))| \leq 4C(j-1)^{-\gamma}$. So

$$A_4 \leq p^d\mathbb{E}(\zeta_1^4) + 48Cdp^d\sum_{j=1}^{p}(2j+1)^{3d-1}(j-1)^{-\gamma} + A_2^2.$$

If $\gamma < 2d$, we get $A_4 = O\left(p^{4d-\gamma}\right)$ so that

$$\sum_{i=0}^{a^d}\mathbb{E}(u_{i,n}-\mathbb{E}u_{i,n})^4 = O(n^{(1-\alpha)d}n^{\alpha(4d-\gamma)}n^{-2d}) = O(n^{(3\alpha-1)d-\gamma\alpha}).$$

In this case we choose, $\alpha < d/(3d-\gamma)$. If $\gamma \geq 2d$, we get $A_4 = O\left(p^{2d}\right)$ so that

$$\sum_{i=0}^{a^d}\mathbb{E}(u_{i,n}-\mathbb{E}u_{i,n})^4 = O(n^{(1-\alpha)d}n^{\alpha(2d)}n^{-2d}) = O(n^{(\alpha-1)d}).$$

In this case any $\alpha < 1$ is convenient.
To check condition 2, we show that for $w\in\{0,\dots,a\}^d$, the variance of the $u_{w,n}$ tends to $\sum_{t=1}^{\ell}\sum_{s=1}^{\ell}\lambda_s\lambda_t\Sigma_{s,t}$ where Σ is defined in Theorem 4.1. Note that

$$u_{w,n} = \frac{p^{d/2}}{n^{d/2}}\frac{1}{p^{d/2}}\sum_{m_i\in P_{w,n}}\sum_{t=1}^{\ell}\lambda_t\zeta_i(r_t).$$

Then

$$\mathrm{Var}\,(u_{w,n}) = \frac{p^d}{n^d}\frac{1}{p^d}\sum_{t=1}^{\ell}\sum_{s=1}^{\ell}\lambda_s\lambda_t \sum_{m_i\in P_{w,n}}\sum_{m_j\in P_{w,n}} \mathrm{Cov}\,(\zeta_i(r_s),\zeta_j(r_t)).$$

As p tends to infinity,

$$\frac{1}{p^d}\sum_{m_i\in P_{w,n}}\sum_{m_j\in P_{w,n}} \mathrm{Cov}\,(\zeta_i(r_s),\zeta_j(r_t)) \longrightarrow \Sigma_{s,t},$$

and pa tends to n so that

$$\sum_{w\in\{0,\ldots,a\}^d} \mathrm{Var}\; u_{w,n} \longrightarrow \sum_{t=1}^{\ell}\sum_{s=1}^{\ell}\lambda_s\lambda_t\Sigma_{s,t}.$$

To check condition 3, we use the weak dependence property. We order the indices w in $\{0,\ldots,a\}^d$ by the lexicographic order from $i=1$ to a^d

$$T(n) \le \sum_{i=2}^{a^d} i p^d \eta(q-2r_\ell) \le a^{2d} p^d \eta(q) = O(n^{2d-\alpha d-\gamma\beta}).$$

In order to find a convenient $\beta<\alpha$, it is necessary that $2d-\alpha d-\gamma\alpha<0$, that is $\alpha>2d/(d+\gamma)$; this is compatible with the condition $\alpha<d/(3d-\gamma)$ only when $\gamma>5d/3$. Then we can find a $\beta<\alpha$ such that $2d-\alpha d-\gamma\beta<0$. From Theorem 4.2, we get that $\sum_{i=1}^{a^d} u_{i,n}$ tends in distribution to $\mathcal{N}(0,\Lambda^t\Sigma\Lambda)$. We now prove that v_n is negligible because its variance tends to 0. We split Q_n into cubic grids of side q. Assume for simplicity sake that n is divisible by q, to avoid to have small remainder parts that do not make any difference in the result. Consider the shape of Q_n. The cubic grids are located in n/p hyperplanes in each of the d dimensions. Each hyperplane contains $(n/q)^{d-1}$ such grids so that there are at most $Q=d(n^d/pq^{d-1})$ cubic grids in Q_n. For each of these grids, give an index $i\in I$ and denote $\zeta_i^q(r)$ the sum of the indicator functions over the grid points that it contains.

$$v_n = \frac{1}{n^{d/2}}\sum_{i\in I}\sum_{s=1}^{\ell}\zeta_i^q(r_s).$$

so that

$$\mathrm{Var}\,(v_n) \le \frac{1}{n^d}\sum_{(i,j)\in I^2}\left|\mathrm{Cov}\left(\sum_{s=1}^{\ell}\lambda_s\zeta_i^q(r_s),\sum_{t=1}^{\ell}\lambda_t\zeta_j^q(r_t)\right)\right|.$$

We define a partition $(G_r)_{r=0,\ldots,[n/q]-1}$ in I^2 by considering the distance rq between the two corresponding cubes C_i^q and C_j^q. Because of the geometrical structure of $Q(n)$, there are at most $Qd3^{d-1}$ elements in G_0 and for each of them we use the Cauchy Schwarz inequality and the bound

$$\frac{1}{n^d}\mathrm{Var}\left(\sum_{t=1}^{\ell}\lambda_t\zeta_i^q(r_t)\right)\approx\frac{q^d}{p^d}\mathrm{Var}\left(u_{i,n}\right)=O\left(\frac{q^d}{n^d}\right),$$

to get

$$\frac{1}{n^d}\sum_{(i,j)\in G_0}\left|\mathrm{Cov}\left(\sum_{s=1}^{\ell}\lambda_s\zeta_i^q(r_s),\sum_{t=1}^{\ell}\lambda_t\zeta_j^q(r_t)\right)\right|=O\left(Q\frac{q^d}{n^d}\right)=O\left(\frac{q}{p}\right).$$

There are at most $Qd(2r+1)^{d-1}$ elements in G_r and for each of them we use the weak dependence inequality:

$$\left|\mathrm{Cov}\left(\sum_{s=1}^{\ell}\lambda_s\zeta_i^q(r_s),\sum_{t=1}^{\ell}\lambda_t\zeta_i^q(r_t)\right)\right|\leq Cq^{2d}(rq)^{-\gamma},$$

so that

$$\begin{aligned}\frac{1}{n^d}\sum_{r=1}^{[n/q]-1}\sum_{(i,j)\in G_r}\left|\mathrm{Cov}\left(\sum_{s=1}^{\ell}\lambda_s\zeta_i^q(r_s),\sum_{t=1}^{\ell}\lambda_t\zeta_j^q(r_t)\right)\right|&=O\left(Q\frac{q^{2d-\gamma}}{n^d}\right)=O\left(\frac{q^{1+d-\gamma}}{p}\right)\\&=O\left(\frac{q}{p}\right).\end{aligned}$$

Then $\mathrm{Var}\left(v_n\right)$ tends to zero. ∎

5. Conclusion

In this paper we introduce a new definition of dependence for point processes; we show that classical point processes are weak dependent in this sense with natural conditions. Further work is needed in two directions: other processes or other statistics:

- It remains to determine conditions for the third large class of dependent processes - namely the class of Gibbs processes - to be weakly dependent. That should be true for the simplest models as the Strauss repulsive processes because of their Markovian structure.
- Using the same line of proof, but taking into account that the number of terms in the sum is now random, one can obtain the same CLT for the $\hat{G}(r)$ estimator of the nearest neighbor function. A further work is to prove the Central Limit Theorem under weak dependence for the Ripley statistic, that is more used than the empirical empty space function. But the task is more difficult because the Ripley statistic is a U-statistic so that the direct use of theorems for sums of variables is not allowed.

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