A simple proof of the Wirsching-Goodwin representation of integers connected to 1 in the 3x + 1 problem

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Summary. *A proof of the Wirsching-Goodwin representation of integers connected to 1 in the $3x+1$ problem (see [3] and [2]) with elementary mathematics.*

1 Basic elements

1.1 Definitions

Let $n \in \mathbb{N}$.

Direct algorithm

$$T(n) = \begin{cases} 3n+1 & if \quad n \equiv 1 \mod 2 \\ n/2 & if \quad n \equiv 0 \mod 2 \end{cases}$$

Inverse algorithm

$$U(n) = \left\{ 2n \quad and \quad \frac{n-1}{3} \quad if \quad n \equiv 4 \mod 6 \right\}$$

Graph $G(n)$

Let $(n_1, n_2) \in \mathbb{N}^2$. n_1 and n_2 are connected by an edge if $n_1 = T(n_2)$ or $n_2 = T(n_1)$. $G(n)$ is the subset of the integers connected to n.

Conjecture "$3x+1$"

$\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : T^k(n) = 1$. An equivalent assertion is $G(1) = \mathbb{N}$.

2 Restriction to odd integers

2.1 f and h

If the "$3x + 1$" conjecture is true for the odd integers it is also true for the even ones by definition of T. The expressions of T and U restricted to odd terms are the following with n odd:

- T becomes f: $f(n) = (3n+1)2^{-j(3n+1)}$ with $j(3n+1)$ is the power of 2 in the prime factors decomposition of $3n + 1$. f is often called the "Syracuse function".
- U becomes h, see[1]:

$$h(n) = \begin{cases} \emptyset & if \quad n \equiv 0 \mod 3 \\ \frac{n2^k-1}{3}, k = 2, 4, 6... & if \quad n \equiv 1 \mod 3 \\ \frac{n2^k-1}{3}, k = 1, 3, 5... & if \quad n \equiv 2 \mod 3 \end{cases}$$

The proof of the expression of h needs the following lemma.

Lemma 1.

$$2^k \overset{\mod 3}{\equiv} \begin{cases} 2 & if \quad k \quad odd \\ 1 & if \quad k \quad even \end{cases}$$

Proof. $(2^k \equiv 1 \mod 3) \Rightarrow (2^k = 3x + 1$ with $x \in \mathbb{N}) \Rightarrow 2^{k+1} = 3.2x + 2 \Rightarrow (2^{k+1} \equiv 1 \mod 3)$.

$(2^k \equiv 2 \mod 3) \Rightarrow (2^k = 3x + 2$ with $x \in \mathbb{N}) \Rightarrow 2^{k+1} = 3(2x + 1) + 1 \Rightarrow (2^{k+1} \equiv 2 \mod 3)$.

$(2^0 \equiv 1 \mod 3) \Rightarrow (2^{2k} \equiv 1 \mod 3)$ and $(2^{2k+1} \equiv 2 \mod 3)$ □

The expression of h comes from the following:

$$f(n) = (3n + 1)2^{-j(3n+1)} \Rightarrow n = \frac{f(n)2^{j(3n+1)} - 1}{3} \in \mathbb{N} \quad (1)$$

There are 3 cases

- $(f(n) \equiv 0 \mod 3)$ (1) is impossible,
- $(f(n) \equiv 1 \mod 3) \Rightarrow f(n) = 3x + 1$ with $x \in \mathbb{N}$

$$\begin{aligned} \frac{f(n)2^{j(3n+1)} - 1}{3} &= \frac{(3x + 1)2^{j(3n+1)} - 1}{3} \\ &= \frac{3x.2^{j(3n+1)} + 2^{j(3n+1)} - 1}{3} \\ &\in \mathbb{N} \quad if \quad j(3n + 1) \; even \quad (lemma 1) \end{aligned}$$

- $(f(n) \equiv 2 \mod 3) \Rightarrow f(n) = 3x + 2$ with $x \in \mathbb{N}$

$$\begin{aligned} \frac{f(n)2^{j(3n+1)} - 1}{3} &= \frac{(3x+2)2^{j(3n+1)} - 1}{3} \\ &= \frac{3x.2^{j(3n+1)} + 2^{j(3n+1)+1} - 1}{3} \\ &\in \mathbb{N} \quad if \quad j(3n+1)\ odd \quad (lemma1) \end{aligned}$$

2.2 Graph $g(n)$

Let (n_1, n_2) be odd integers. n_1 and n_2 are connected by an edge if $n_1 = f(n_2)$ or $n_1 = f(n_2)$. $g(n)$ is the subset of the odd integers connected to n.

3 Properties of $g(1)$

3.1 Expression of $n \in g(1)$ as a sum of fractions

Theorem 1. *Let $n \in g(1)$. $\exists (b, a > u_1 > u_2, ... > u_b = 0) \in \mathbb{N}^{b+2}$:*

$$n = \frac{2^a}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}}.$$

Note that $\frac{2^a}{3^b} \geq 1 \Rightarrow a \geq b\frac{log3}{log2}$.

Proof. $n \in g(1) \Leftrightarrow \exists b : n \in h^{(b)}(1)$. The proof use induction with b. Theorem 1 is true for $b = 1$ because $h(1) = \left\{ \frac{2^k - 1}{3}, k = 2, 4, 6..., \right\}$, and for $b = 2$ because

$$\begin{aligned} h^{(2)}(1) &\subset \left\{ \frac{1}{3} \left(\frac{2^{k_1} - 1}{3} 2^{k_2} - 1 \right), k_1 = 2, 4, 6...; k_2 \in \mathbb{N} \right\} \\ &\subset \left\{ \frac{2^{k_1+k_2}}{3^2} - \frac{2^{k_2}}{3^2} - \frac{2^0}{3}, k_1 = 2, 4, 6...; k_2 \in \mathbb{N} \right\} \end{aligned}$$

Assume that theorem 1 is true for $l \leq b - 1$.

$$\begin{aligned} h^{(b)}(1) &\subset \left\{ \frac{1}{3} \left[\left(\frac{2^a}{3^{b-1}} - \sum_{i=1,b-1} \frac{2^{u_i}}{3^{b-i}} \right) 2^k - 1 \right], (a > u1 > ...u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\} \\ &\subset \left\{ \frac{2^{a+k}}{3^b} - \sum_{i=1,b-1} \frac{2^{u_i+k}}{3^{b-i+1}} - \frac{2^0}{3}, (a > u1 > ...u_{b-1} = 0, k) \in \mathbb{N}^{b+1} \right\} \end{aligned}$$

The last expression has the form claimed in theorem 1. □

Note that k_1 is even but k_l may be odd or even for $l > 1$. Thus $a - u_1$ is even. If $a - u_1 = 2$, $h(1) = 1$, so the first "interesting" value is $a - u_1 = 4$.

Proof. An alternative proof of theorem 1 using $f : n \in g(1) \Leftrightarrow \exists b \in \mathbb{N} : f^b(n) = 1$. b is the number of odd integers (excluding 1) in the sequence from n to 1.

Induction with b:

Let $b = 1$. $3n + 1 = 2^{j(3n+1)}x$ a partial prime factors decomposition of $3n + 1$.

$$\begin{aligned} f(n) &= (3n+1)2^{-j(3n+1)} \\ &= 2^{j(3n+1)}x2^{-j(3n+1)} \\ &= x \end{aligned}$$

$$b = 1 \Rightarrow f(n) = 1 \Rightarrow x = 1 \Rightarrow 3n + 1 = 2^{j(3n+1)} \Rightarrow n = \frac{2^{j(3n+1)}}{3} - \frac{1}{3}.$$

Let $b = 2$. $b = 2 \Rightarrow f(f(n)) = 1 \Rightarrow f(n) = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3}$.

$$f(n) = (3n+1)2^{-j(3n+1)} \Rightarrow (3n+1)2^{-j(3n+1)} = \frac{2^{j(3f(n)+1)}}{3} - \frac{1}{3}.$$

Therefore

$$n = \frac{2^{j(3n+1)+j(3f(n)+1}}{3^2} - \frac{2^{j(3n+1)}}{3^2} - \frac{1}{3}.$$

Assuming that the theorem is true till $b - 1$ we have to prove that it is true for b.

$$f(n) = (3n+1)2^{-j(3n+1)} = \frac{2^a}{3^{b-1}} - \sum_{i=1,b-1} \frac{2^{u_i}}{3^{b-i}}.$$

Therefore

$$n = \frac{2^{a+j(3n+1)}}{3^b} - \sum_{i=1,b-1} \frac{2^{u_i+j(3n+1)}}{3^{b-i+1}} - \frac{1}{3}.$$

□

Note that the general form of u_i is thus $u_{b-i} = \sum_{l=1,i} j[3f^{(l-1)}(n)+1]$, with $f^{(0)} = Id$, and $a = \sum_{l=1,b} j[3f^{(l-1)}(n) + 1]$.

3.2 Admissible tuple $(b, a > u_1 > u_2, ... > u_b = 0)$

Only some values of $(b, a > u_1 > u_2, ... > u_b = 0)$ give an integer n in theorem 1, most of them do not.

Definition 1. *A tuple* $(b, a \geq b\frac{log3}{log2}, a > u_1 > u_2, ... > u_b = 0)$ *of* $b + 1$ *integers is admissible if* $\frac{2^a}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}} \in \mathbb{N}$.

The admissible parity of $u_i - u_{i+1}$ is determined by the remainder modulo 3 of the integer obtained at step i (see the definition of h in section 2.1).

Lemma 2. *Let* $\frac{n2^k-1}{3} \in \mathbb{N}$ *and* $\frac{n2^k-1}{3} \equiv v \mod 3$. *Then* $\frac{n2^{k+2}-1}{3} \equiv v + 1 \mod 3$.

The lemma indicates that v is a periodic function of k with period 6: $\frac{n2^k-1}{3} \equiv \frac{n2^{k+6}-1}{3}$ mod 3.

Proof. $\frac{n2^k-1}{3} \equiv v \mod 3 \Rightarrow \frac{n2^k-1}{3} = 3x + v$,

$$\begin{aligned} \frac{n2^{k+2}-1}{3} &= 4\frac{n2^k-1}{3} + 1 \\ &= 4(3x+v)+1 \\ &\equiv v+1 \mod 3 \end{aligned}$$

□

Lemma 3. *Let $n \in \mathbb{N}$ and $n_1 = \frac{n2^k-1}{3} \notin \mathbb{N}$. Then $\forall l \in \mathbb{N}$, $\quad \frac{n_1 2^l-1}{3} \notin \mathbb{N}$.*

The lemma indicates that if $(b, a > u_1 > u_2, ... > u_b = 0)$ is admissible and k has not the correct parity, the tuple $(b+1, a+k > u_1+k > u_2+k, .. > u_b+k, u_{b+1} = 0)$ is not admissible and all tuples based on it are also not admissible. Conversely, if $(b, a > u_1 > u_2, ... > u_b = 0)$ is admissible, $(b-1, a-u_{b-1} > u_1-u_{b-1} > u_2-u_{b-1}, ... > u_{b-1}-u_{b-1} = 0)$ is also admissible and all such successive reduced tuples till $(1, a-u_1 > u_1-u_1 = 0)$

Proof. $n_1 = \frac{n2^k-1}{3} = \frac{p}{3}$, with p and 3 relatively prime. $\frac{n_1 2^l-1}{3} = \frac{\frac{p}{3}2^l-1}{3} = \frac{p2^l-3}{9}$. Suppose that $\frac{p2^l-3}{9} = x \in \mathbb{N}$. Then $p2^l = 9x+3$ that is impossible because p and 3 are relatively prime. □

3.3 Tree structure of $g(1)$

Lemma 4. *$g(1)$ is a tree with an additionnal loop in its root 1.*

Proof. Let h^* be a modified version of h: $h^*(1) = \frac{n2^k-1}{3}, k = 4, 6, 8...$,

$g^*(1) = \{1 \cup h^*(1) \cup h[h^*(1)], .. \cup h^{(l)}[h^*(1)]..\}$. The case $n \in h(n_1) \cap h(n_2)$ with $n_1 \neq n_2$, is impossible because there is only one $f(n)$. Thus $g^*(1)$ is a tree because any $n \in g^*(1)$ cannot have two different parents. $g(1)$ is equal to $g^*(1)$ with a supplementary loop at node 1. □

The following definition 2 and proposition 1 are not nessessary for the proof of theorem 3 and may be skipped.

Definition 2. *$g^*(1)[t, s] \subset g^*(1)$ is the graph generated by the admissible tuples with $b \leq t$, $4 \leq a - u_1 \leq 2 + 6s$ and $u_i - u_{i+1} \leq 6s$.*

Proposition 1. *$|g^*(1)[t, s]| = 1 + \frac{3s[(2s)^t-1]}{2s-1}$, with $|A|$ is the cardinal of the set A.*

Proof. Lemma 2 implies that for each node of the tree there are $3s$ admissible children of which $2s$ have children. □

Note that with $s = 1$ one obtains that the ratio of integers pertaining to $g^*(1)[t, 1]$ and less than $\max(g^*(1)[t, 1]) \simeq \frac{2^{2+6t}}{3^t}$ is greater than $\frac{3}{4}\left(\frac{2^5}{3}\right)^t$.

Lemma 5. $2^{2.3^k} - 1 \equiv 0 \mod 3^{k+1}$

Proof. By induction. The lemma is true for $k = 0$ and $k = 1$ because $2^2 - 1 = 3$ and $2^6 - 1 = 63 = 3^2 7$. Assume that lemma is true till $k - 1$, with $\left(2^{2.3^{k-1}} - 1\right) = 3^k x$ with $x \in \mathbb{N}$.

$$\begin{aligned}
2^{2.3^k} - 1 &= \left(2^{2.3^{k-1}}\right)^3 - 1 \\
&= \left(2^{2.3^{k-1}} - 1\right)\left[\left(2^{2.3^{k-1}}\right)^2 + \left(2^{2.3^{k-1}}\right) + 1\right] \\
&= 3^k x \left[\left(2^{2.3^{k-1}}\right)^2 - 1 + \left(2^{2.3^{k-1}} - 1\right) + 3\right] \\
&= 3^k x \left[\left(2^{2.3^{k-1}} - 1\right)\left(2^{2.3^{k-1}} + 1\right) + 3^k x + 3\right] \\
&= 3^k x \left[3^k x \left(2^{2.3^{k-1}} + 1\right) + 3^k x + 3\right] \\
&= 3^{k+1} x \left[3^{k-1} x \left(2^{2.3^{k-1}} + 2\right) + 1\right]
\end{aligned}$$

□

Lemma 6. *Let* $(b, u_0 > u_1 > u_2, ... > u_b = 0)$ *an admissible tuple. Let* $j < b$ *and* $u'_i = u_i + 2.3^{b-j-1}$ *if* $i \leq j$ *and* $u'_i = u_i$ *if* $i > j$*. Then the tuple* $(b, u'_0 > u'_1 > u'_2, ... > u'_b = 0)$ *is admissible.*

Proof. Let $n = \frac{2^{u_0}}{3^b} - \sum_{i=1,b} \frac{2^{u_i}}{3^{b-i+1}}$, and $x = \frac{2^{u'_0}}{3^b} - \sum_{i=1,b} \frac{2^{u'_i}}{3^{b-i+1}}$.

$$\begin{aligned}
x - n &= \left(\frac{2^{u_0}}{3^b} - \sum_{i=1,j} \frac{2^{u_i}}{3^{b-i+1}}\right)\left(2^{2.3^{b-j-1}} - 1\right) \\
&= \left(\frac{2^{u_0}}{3^b} - \sum_{i=1,j} \frac{2^{u_i}}{3^{b-i+1}}\right)\left(q 3^{b-j}\right) \\
&= q\left(\frac{2^{u_0}}{3^j} - \sum_{i=1,j} \frac{2^{u_i}}{3^{j-i+1}}\right)
\end{aligned}$$

Lemma 5 implies that $q \in \mathbb{N}$ and lemma 3 implies that the second term is integer, therefore x is integer. □

We introduce an alternative notation for the tuple $(b, u_0 > u_1 > u_2, ... > u_b = 0)$.

Let $v_i = u_{i-1} - u_i,\ i = 1, ...b$. The tuple $(b, \sum_{i=1,b} v_i, \sum_{i=2,b} v_i, ...v_b)$ is equal to the tuple $(b, u_0 > u_1 > u_2, ... > u_b = 0)$. The alternative notation for this tuple is $(b, v_1, v_2, ..., v_b)$.

Note that $v_i = j(3f^{(b-i)}(n) + 1)$, with n given by theorem1, see the second proof of theorem1.

Theorem 2. *Let $v_i \in \mathbb{N},\ i = 2, ...b$ with $1 \leq v_i \leq 2.3^{b-i}$ and $b > 1$. For each tuple $(v_2, v_3, ...v_b)$ $\exists v_1$ even with $4 \leq v_1 \leq 2.3^{b-1}$ such that $(b, \sum_{i=1,b} v_i, \sum_{i=2,b} v_i, ...v_b)$ is admissible.*

Proof. On one side the cardinal number of $\{v_2, ...v_b\}$ is

$$\begin{aligned} |\{v_2, ...v_b\}| &= \prod_{i=2,b} 2.3^{b-i} \\ &= 2^{b-1} 3^{\sum_{i=2,b}(b-i)} \\ &= 2^{b-1} 3^{\sum_{k=0,b-2} k} \\ &= 2^{b-1} 3^{\frac{(b-2)(b-1)}{2}} \end{aligned}$$

On the other side the number of admissible $\{v_1 v_2, ...v_b\}$ nodes of $g^*(1)$ is equal to the product of the number of admissible nodes with children at each step excepted the last one with sterile nodes taken into account. At the first step v_1, this number is $\frac{2}{3}3^{b-1}$, at the step v_2 it is equal to $\frac{2}{3}3^{b-2}$, and so on till the last step with one admissible node (with or without child for this last step). The product is equal to $2^{b-1}3^{\frac{(b-2)(b-1)}{2}}$. The admissible nodes are different from each other because $g^*(1)$ is a tree, therefore there is a one to one function between the admissible nodes obtained with b steps and the tuple $\{v_1^*, v_2, ...v_b\}$ defined in the theorem. □

Example.

v1*	4	4	8	8	10	10	14	14	16	16	20	20
v2	3	5	2	6	1	5	4	6	1	3	2	4
v3	2	1	1	2	1	2	2	1	2	1	2	1
n	17	35	75	2417	151	4849	...					1242755

Table 1: The 12 admissible tuples with $b = 3$

Among the possible sequences $(v_2, ...v_b)$ allowed by the Theorem 2 some are specially interesting such as the strictly ascending sequence $(f^{(i)}(n),\ i = 1, b-1)$ (see $n = 151$ in the above table as an example), given in the following corollary.

Corollary. $\forall b \in \mathbb{N},\ \exists n \in \mathbb{N}\ : \forall i \in (1, b-1),\ f^{(i)}(n) > f^{(i-1)}(n)$.

Proof. n is obtained with $v_i = 1,\ i = 2 : b$, and v_1 given by theorem 2 and lemma 6. □

The Wirsching-Goodwin representation of the nodes of $g(1)$ obtained with b steps (see [2]) may be now stated in the following theorem. Let $g^*(1, b) = \{n \in g(1)\ :\ f^{(b)} = 1$ and $f^{(b-1)} \neq 1\}$ and v_1^* the value of v_1 whose existence is proven in theorem 2.

Theorem 3. *There is a one to one relation between $g^*(1, b)$ with $b > 1$ and the set of the tuples $(b, v_1', v_2', ..., v_b')$ with $v_i' = v_i + 2.3^{b-i}c_i,\ c_i \in \mathbb{N}^*,\ v_i \in \mathbb{N},\ i = 2, ...b$ with $1 \leq v_i \leq 2.3^{b-i}$ and $4 \leq v_1 = v_1^* \leq 2.3^{b-1}$.*

Proof. Direct from theorem2 and lemma 6. □

References

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